New Structures Based on Completions

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Abstract. We propose new axioms relative to combinatorial topology. These axioms are settled in the framework of completions which are inductive properties expressed in a declarative way, and that may be combined.
We introduce several completions for describing *dyads*. A dyad is a pair of complexes which are, in a certain sense, linked by a "relative topology". We first give some basic properties of dyads, then we introduce a second set of axioms for *relative dendrites*. This allows us to establish a theorem which provides a link between dyads and dendrites, a dendrite is an acyclic complex which may be also described by completions. Thanks to a previous result, this result makes clear the relation between dyads, relative dendrites, and complexes which are acyclic in the sense of homology.

Keywords: Acyclic complexes, Combinatorial topology, Simplicial Complexes, Collapse, Completions.

1 Introduction

Simple homotopy plays a fundamental role in combinatorial topology [1–7]. It has also been shown that the collapse operation is fundamental to interpret some notions relative to homotopy in the context of computer imagery [8–10], see also [11–13].

In this paper, we further investigate an axiomatic approach related to simple homotopy. This approach has been introduced in [14] where the notion of a *dendrite* was presented through two simple axioms. A dendrite is an acyclic object. A theorem asserts that an object is a dendrite if and only if it is acyclic in the sense of homology.

Here, we present new axioms for describing *dyads*. Intuitively, a dyad is a couple of objects (X, Y), with $X \subseteq Y$, such that the cycles of X are "at the right place with respect to the ones of Y". Let us consider Fig. 1, where an object X, and two objects $Y \subseteq X$, $Z \subseteq X$ are depicted. We see that it is possible to continuously deform Y onto X, this deformation keeping Y inside X. Thus, the pair (Y, X) is a dyad. On the other hand, Z is homotopic to X, but Z is not "at the right place", therefore (Z, X) is not a dyad.

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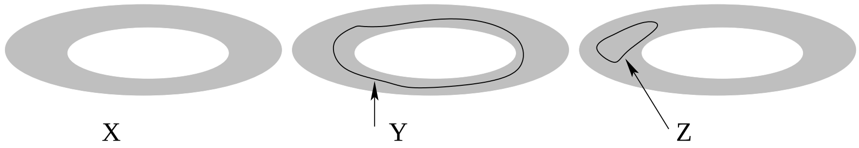

Fig. 1. An object X (an annulus), and two objects $Y \subseteq X$, $Z \subseteq X$ (two simple closed curves). The pair (Y, X) is a dyad, while (Z, X) is not.

The paper is organized as follows. First, we give some basic definitions for simplicial complexes (Sec. 2). Then, we recall some basic facts relative to the notion of a completion (Sec. 3), completions will be used as a language for describing our axioms. We also recall the definition of a dendrite (Sec. 4). In the two following sections we introduce new axioms for presenting the notion of a dyad (Sec. 5), and the notion of a relative dendrite (Sec. 6). In Sec. 7, we give a theorem (Th. 4) which makes clear the link between dyads and dendrites. Thanks to a previous result, this result makes clear the relation between dyads, relative dendrites, and complexes which are acyclic in the sense of homology.

The paper is self contained. In particular, almost all proofs are included.

2 Basic definitions for simplicial complexes

Let X be a finite family composed of finite sets. The *simplicial closure of* X is the complex $X^- = \{y \subseteq x \mid x \in X\}$. The family X is a *(finite simplicial) complex* if $X = X^-$. We write $\mathbb{S}$ for the collection of all finite simplicial complexes. Note that $\emptyset \in \mathbb{S}$ and $\{\emptyset\} \in \mathbb{S}$, $\emptyset$ is the *void complex*, and $\{\emptyset\}$ is the *empty complex*.

Let $X \in \mathbb{S}$. An element of X is *a simplex of* X or *a face of* X. A *facet of* X is a simplex of X which is maximal for inclusion.

A *simplicial subcomplex* of $X \in \mathbb{S}$ is any subset Y of X which is a simplicial complex. If Y is a subcomplex of X, we write $Y \preceq X$.

Let $X \in \mathbb{S}$. The *dimension* of $x \in X$, written $dim(x)$, is the number of its elements minus one. The *dimension of* X, written $dim(X)$, is the largest dimension of its simplices, the *dimension of* $\emptyset$ is defined to be -1.

A complex $A \in \mathbb{S}$ is *a cell* if $A = \emptyset$ or if A has precisely one non-empty facet x. We set $A^\circ = A \setminus \{x\}$ and $\emptyset^\circ = \emptyset$. We write $\mathbb{C}$ for the collection of all cells. A cell $\alpha \in \mathbb{C}$ is *a vertex* if $dim(\alpha) = 0$.

The *ground set* of $X \in \mathbb{S}$ is the set $\underline{X} = \cup\{x \in X \mid dim(x) = 0\}$. We say that $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are *disjoint*, or that X is *disjoint from* Y, if $\underline{X} \cap \underline{Y} = \emptyset$. Thus, X and Y are disjoint if and only if $X \cap Y = \emptyset$ or $X \cap Y = \{\emptyset\}$.

If $X \in \mathbb{S}$ and $Y \in \mathbb{S}$ are disjoint, the *join of* X *and* Y is the simplicial complex XY such that $XY = \{x \cup y \mid x \in X, y \in Y\}$. Thus, $XY = \emptyset$ if $Y = \emptyset$ and $XY = X$ if $Y = \{\emptyset\}$. The join αX of a vertex α and a complex $X \in \mathbb{S}$ is a *cone*.

Important convention. In this paper, if $X, Y \in \mathbb{S}$, we implicitly assume that X and Y have disjoint ground sets whenever we write XY.

Let $A \in \mathbb{C}$ and $X \preceq A$. The *dual of* X *for* A is the simplicial complex, written X^*_A, such that $X^*_A = \{x \in A \mid (\underline{A} \setminus x) \notin X\}$.
We have $\emptyset^*_A = A$ and $\{\emptyset\}^*_A = A^\circ$, and, for any $A \in \mathbb{C}$, we have the following:
- If $X \preceq A$, then $(X^*_A)^*_A = X$.
- If $X \preceq A$, $Y \preceq A$, then $(X \cup Y)^*_A = X^*_A \cap Y^*_A$ and $(X \cap Y)^*_A = X^*_A \cup Y^*_A$.

3 Completions

We give some basic definitions for completions, they will allow us to formulate our axioms as well as to combine them. A completion may be seen as a rewriting rule which permits to derive collections of sets. See [14] for more details.

Let $\mathbf{S}$ be a given collection and let $\mathcal{K}$ be an arbitrary subcollection of $\mathbf{S}$. Thus, we have $\mathcal{K} \subseteq \mathbf{S}$. In the sequel of the paper, the symbol $\mathcal{K}$, with possible superscripts, will be a dedicated symbol (a kind of variable).

Let $\boldsymbol{\kappa}$ be a binary relation on $2^{\mathbf{S}}$, thus $\boldsymbol{\kappa} \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that $\boldsymbol{\kappa}$ is *finitary*, if $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$.

Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$. We say that $\langle \mathrm{K} \rangle$ is a *completion (on* $\mathbf{S}$*)* if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
-> If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$. $\langle \boldsymbol{\kappa} \rangle$
where $\boldsymbol{\kappa}$ is an arbitrary finitary binary relation on $2^{\mathbf{S}}$.

If $\langle \mathrm{K} \rangle$ is a property which depends on $\mathcal{K}$, we say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.

Theorem 1. [14] *Let* $\langle \mathrm{K} \rangle$ *be a completion on* $\mathbf{S}$ *and let* $\mathbf{X} \subseteq \mathbf{S}$. *There exists, under the subset ordering, a unique minimal collection which contains* $\mathbf{X}$ *and which satisfies* $\langle \mathrm{K} \rangle$.

If $\langle \mathrm{K} \rangle$ is a completion on $\mathbf{S}$ and if $\mathbf{X} \subseteq \mathbf{S}$, we write $\langle \mathbf{X}; \mathrm{K} \rangle$ for the unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle$.

Let $\langle \mathrm{K} \rangle$ be a completion which is expressed as the above property $\langle \boldsymbol{\kappa} \rangle$. By a fixed point property, the collection $\langle \mathbf{X}; \mathrm{K} \rangle$ may be obtained by starting from $\mathcal{K} = \mathbf{X}$, and by iteratively adding to $\mathcal{K}$, until idempotence, all the sets $\mathbf{G}$ such that $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$ and $\mathbf{F} \subseteq \mathcal{K}$ (see [14]).

Let $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ be two completions on $\mathbf{S}$. It may be seen that $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion, the symbol $\wedge$ standing for the logical "and". In the sequel of the paper, we write $\langle \mathrm{K}, \mathrm{Q} \rangle$ for $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$. Also, if $\mathbf{X} \subseteq \mathbf{S}$, the notation $\langle \mathbf{X}; \mathrm{K}, \mathrm{Q} \rangle$ stands for the smallest collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$.

Example. Let us consider the collection $\mathbf{S} = \mathbb{S}$. Thus, $\mathcal{K}$ denotes an arbitrary collection of simplicial complexes. We define the property $\langle \Upsilon \rangle$ as follows:
-> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$
Let $\boldsymbol{\kappa}$ be the binary relation on $2^{\mathbf{S}}$ such that $(\mathbf{F}, \mathbf{G}) \in \boldsymbol{\kappa}$ iff there exist $S, T \in \mathbb{S}$, with $\mathbf{F} = \{S, T\}$, $\mathbf{G} = \{S \cup T\}$, and $S \cap T \neq \{\emptyset\}$. We see that $\boldsymbol{\kappa}$ is finitary and that $\langle \Upsilon \rangle$ may be expressed as the property $\langle \boldsymbol{\kappa} \rangle$. Thus $\langle \Upsilon \rangle$ is a completion. Now, let us consider the collection $\Pi = \langle \mathbb{C}; \Upsilon \rangle$. It may be checked that Π is

precisely the collection of all simplicial complexes which are (path) connected (see also [17] where the property $\langle \Upsilon \rangle$ is used in a different context). Having in mind the above iterative procedure, $\langle \mathbb{C}, \Upsilon \rangle$ may be seen as a dynamic structure where the completion $\langle \Upsilon \rangle$ acts as a generator, which, from $\mathbb{C}$, makes it possible to enumerate all finite connected simplicial complexes.

4 Dendrites

The notion of a dendrite was introduced in [14] as a way for defining a remarkable collection made of acyclic complexes.

In the rest of the paper, $\mathcal{K}$ will denote an arbitrary subcollection of $\mathbb{S}$.

Definition 1. *We define the two completions on* $\mathbb{S}$*: For any* $S, T \in \mathbb{S}$*,*

-> *If* S*,* $T \in \mathcal{K}$*, then* $S \cup T \in \mathcal{K}$ *whenever* $S \cap T \in \mathcal{K}$*.* $\langle \mathrm{D1} \rangle$

-> *If* S*,* $T \in \mathcal{K}$*, then* $S \cap T \in \mathcal{K}$ *whenever* $S \cup T \in \mathcal{K}$*.* $\langle \mathrm{D2} \rangle$

We set $\mathbb{R} = \langle \mathbb{C}; \mathrm{D1} \rangle$ *and* $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$*, thus we have* $\mathbb{R} \subseteq \mathbb{D}$*.*
Each element of $\mathbb{R}$ *is a* ramification *and each element of* $\mathbb{D}$ *is a* dendrite*.*

Let us recall some basic definitions relative to simple homotopy [1], note that these notions may also be introduced by the means of completions [14].

Let $X, Y \in \mathbb{S}$ and x, y be two distinct faces of X. If y is the only face of X which contains x, then $Y = X \setminus \{x, y\}$ is *an elementary collapse* of X. We say that X *collapses onto* Y, if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and X_i is an elementary collapse of X_{i-1}, $i \in [1, k]$. The complex X is *collapsible* if X collapses onto $\emptyset$. We say that X is *(simple) homotopic to* Y if there exists a sequence $\langle X_0, ..., X_k \rangle$ such that $X_0 = X$, $X_k = Y$, and either X_i is an elementary collapse of X_{i-1}, or X_{i-1} is an elementary collapse of X_i, $i \in [1, k]$. The complex X is *(simply) contractible* if X is simple homotopic to $\emptyset$.

For example, if X is a tree, then X is collapsible, X is a dendrite, and also a ramification. In fact, any collapsible complex is a ramification [6]. The Bing's house with two rooms [15] and the dunce hat [16] are classical examples of complexes which are contractible but not collapsible. Both of them are dendrites.

In fact, it was shown [14] that any simply contractible complex is a dendrite. Furthermore it was shown that:
- a complex is a dendrite if and only if it is acyclic in the sense of homology; and
- a complex is a dendrite if and only if its suspension is simply contractible.

5 Dyads

In this section, we introduce the notion of a dyad and give some propositions which are necessary to establish one of the main result of the paper (Th. 4). See the introduction and Fig. 1 for an intuitive presentation of a dyad. See also Fig. 2 for an illustration of the axiom $\langle \ddot{\mathrm{X}}1 \rangle$.

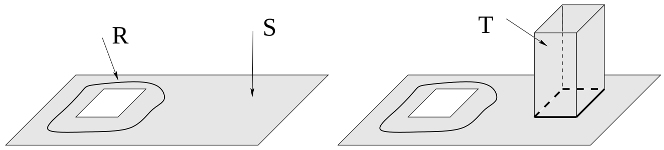

Fig. 2. Two objects R, S which constitute a dyad (R, S). An object T which is glued to S. The couple $(S \cap T, T)$ is a dyad, thus, by axiom $\langle \ddot{\mathrm{X}}1 \rangle$, $(R, S \cup T)$ is also a dyad.

We set $\ddot{\mathbb{S}} = \{(X, Y) \mid X, Y \in \mathbb{S}, X \preceq Y\}$ and $\ddot{\mathbb{C}} = \{(A, B) \in \ddot{\mathbb{S}} \mid A, B \in \mathbb{C}\}$. In the sequel of the paper, $\ddot{\mathcal{K}}$ will denote an arbitrary subcollection of $\ddot{\mathbb{S}}$. Furthermore, α and β will always denote vertices.

Definition 2. *We define three completions on* $\ddot{\mathbb{S}}$*: For any* $(R, S) \in \ddot{\mathbb{S}}$, $T \in \mathbb{S}$,

-> *If* $(R, S) \in \ddot{\mathcal{K}}$ *and* $(S \cap T, T) \in \ddot{\mathcal{K}}$*, then* $(R, S \cup T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}1 \rangle$
-> *If* $(R, S) \in \ddot{\mathcal{K}}$ *and* $(R, S \cup T) \in \ddot{\mathcal{K}}$*, then* $(S \cap T, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}2 \rangle$
-> *If* $(R, S \cup T) \in \ddot{\mathcal{K}}$ *and* $(S \cap T, T) \in \ddot{\mathcal{K}}$*, then* $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}3 \rangle$

We set $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{X}}1, \ddot{\mathrm{X}}2, \ddot{\mathrm{X}}3 \rangle$. *Each element of* $\ddot{\mathbb{X}}$ *is a* dyad.

We introduce the following completions on $\ddot{\mathbb{S}}$ (the symbols $\ddot{\mathrm{T}}$, $\ddot{\mathrm{U}}$, $\ddot{\mathrm{L}}$ stand respectively for "transitivity", "upper confluence", and "lower confluence"):
For any $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$,

-> If $(R, S) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{T}} \rangle$
-> If $(R, S) \in \ddot{\mathcal{K}}$ and $(R, T) \in \ddot{\mathcal{K}}$, then $(S, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{U}} \rangle$
-> If $(R, T) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{L}} \rangle$

Considering complexes R, S, T such that $R \preceq S \preceq T$, we see that we obtain directly $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, $\langle \ddot{\mathrm{L}} \rangle$ from $\langle \ddot{\mathrm{X}}1 \rangle$, $\langle \ddot{\mathrm{X}}2 \rangle$, $\langle \ddot{\mathrm{X}}3 \rangle$, respectively. Thus, we have:

Proposition 1. *The collection* $\ddot{\mathbb{X}}$ *satisfies the properties* $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, *and* $\langle \ddot{\mathrm{L}} \rangle$.

Proposition 2. *For any* $X \in \mathbb{S}$, *we have* $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$.

Proof. If $X = \emptyset$, then $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$ (since $(\emptyset, \emptyset) \in \ddot{\mathbb{C}}$). If $X = \{\emptyset\}$, then $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$ (since $\alpha X = \alpha$ and $(\emptyset, \alpha) \in \ddot{\mathbb{C}}$). Suppose $X \neq \emptyset$ and $X \neq \{\emptyset\}$.
i) If X has a single facet, then $X \in \mathbb{C}$. Thus $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$ (since $(\emptyset, \alpha X) \in \ddot{\mathbb{C}}$);
ii) If X has more than one facet, then there exists $X', X'' \in \mathbb{S}$ such that $X = X' \cup X''$, and $Card(X') < Card(X)$, $Card(X'') < Card(X)$. Suppose that $(\emptyset, \alpha X') \in \ddot{\mathbb{X}}$, $(\emptyset, \alpha X'') \in \ddot{\mathbb{X}}$, and $(\emptyset, \alpha(X' \cap X'')) \in \ddot{\mathbb{X}}$. Then, by $\langle \ddot{\mathrm{U}} \rangle$, we have $(\alpha(X' \cap X''), \alpha X'') \in \ddot{\mathbb{X}}$. Therefore, by $\langle \ddot{\mathrm{X}}1 \rangle$ (setting $R = \emptyset$, $S = \alpha X'$, $T = \alpha X''$), we have $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$. The result follows by induction on $Card(X)$. $\square$

Proposition 3. *For any* $X \in \mathbb{S}$, *we have* $(X, X) \in \ddot{\mathbb{X}}$.

Proof. By Prop. 2, we have $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$. Since $\ddot{\mathbb{X}}$ satisfies $\langle \ddot{\mathrm{U}} \rangle$, it implies that $(\alpha X, \alpha X) \in \ddot{\mathbb{X}}$ (setting $R = \emptyset$, and $S = T = \alpha X$). By $\langle \ddot{\mathrm{X}}2 \rangle$ (setting

$R = S = \alpha X$, and $T = X$), this gives $(\alpha X \cap X, X) = (X, X) \in \ddot{\mathbb{X}}$. $\square$

We define two completions on $\ddot{\mathbb{S}}$: For any S, $T \in \mathbb{S}$,

-> If $(S \cap T, T) \in \ddot{\mathcal{K}}$, then $(S, S \cup T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}1 \rangle$

-> If $(S, S \cup T) \in \ddot{\mathcal{K}}$, then $(S \cap T, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}2 \rangle$

We give, hereafter, a theorem (Th. 2) which provides another way to generate the collection $\ddot{\mathbb{X}}$. This theorem will be used in section 7 to establish a link between dendrites and dyads. Before, we make a remark on a basic property of completions which allows one to establish the equivalence between two completions structures. This property is necessary for the proof of Th. 2.

Remark 1. Let $\langle \mathrm{K} \rangle$ be a completion on $\mathbf{S}$ and let $\mathbf{X} \subseteq \mathbf{S}$. It may be shown [14] that we have $\langle \mathbf{X}; \mathrm{K} \rangle = \cap\{\mathbf{Y} \subseteq \mathbf{S} \mid \mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{Y}$ satisfies $\langle \mathrm{K} \rangle\}$. Thus, if a given collection $\mathbf{Y} \subseteq \mathbf{S}$ is such that $\mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{Y}$ satisfies $\langle \mathrm{K} \rangle$, then we have necessarily $\langle \mathbf{X}; \mathrm{K} \rangle \subseteq \mathbf{Y}$.

Theorem 2. *We have* $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Proof. We set $\ddot{\mathbb{X}}' = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$. As a consequence of Prop. 3, we can obtain $\langle \ddot{\mathrm{Y}}1 \rangle$ and $\langle \ddot{\mathrm{Y}}2 \rangle$ from $\langle \ddot{\mathrm{X}}1 \rangle$ and $\langle \ddot{\mathrm{X}}2 \rangle$, respectively (setting $R = S$). The collection $\ddot{\mathbb{X}}$ also satisfies the properties $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, $\langle \ddot{\mathrm{L}} \rangle$ (Prop. 1). Thus, since $\ddot{\mathbb{C}} \subseteq \ddot{\mathbb{X}}$, we have $\ddot{\mathbb{X}}' \subseteq \ddot{\mathbb{X}}$ (see remark 1). Now, let $(R, S) \in \ddot{\mathbb{S}}$ and $T \in \mathbb{S}$:
- Suppose $(R, S) \in \ddot{\mathbb{X}}'$ and $(S \cap T, T) \in \ddot{\mathbb{X}}'$. Then, by $\langle \ddot{\mathrm{Y}}1 \rangle$, we have $(S, S \cup T) \in \ddot{\mathbb{X}}'$. Therefore, by $\langle \ddot{\mathrm{T}} \rangle$, we have $(R, S \cup T) \in \ddot{\mathbb{X}}'$,
- Suppose $(R, S) \in \ddot{\mathbb{X}}'$ and $(R, S \cup T) \in \ddot{\mathbb{X}}'$. Then, by $\langle \ddot{\mathrm{U}} \rangle$, we have $(S, S \cup T) \in \ddot{\mathbb{X}}'$. Therefore, by $\langle \ddot{\mathrm{Y}}2 \rangle$, we have $(S \cap T, T) \in \ddot{\mathbb{X}}'$,
- Suppose $(R, S \cup T) \in \ddot{\mathbb{X}}'$ and $(S \cap T, T) \in \ddot{\mathbb{X}}'$. Then, by $\langle \ddot{\mathrm{Y}}1 \rangle$, $(S, S \cup T) \in \ddot{\mathbb{X}}'$. Therefore, by $\langle \ddot{\mathrm{L}} \rangle$, we have $(R, S) \in \ddot{\mathbb{X}}'$.

It follows that $\ddot{\mathbb{X}}'$ satisfies the three properties $\langle \ddot{\mathrm{X}}1 \rangle$, $\langle \ddot{\mathrm{X}}2 \rangle$, $\langle \ddot{\mathrm{X}}3 \rangle$. Thus, since $\ddot{\mathbb{C}} \subseteq \ddot{\mathbb{X}}'$, we have $\ddot{\mathbb{X}} \subseteq \ddot{\mathbb{X}}'$ (see remark 1). $\square$

6 Relative dendrites

In this section, we introduce new axioms for defining the notion of a relative dendrite. We will see in the sequel (next section) that these axioms provide another way to describe dyads. We set $\ddot{\mathbb{C}}^+ = \ddot{\mathbb{C}} \cup \{(\{\emptyset\}, \{\emptyset\})\}$.

Definition 3. *We define two completions on* $\ddot{\mathbb{S}}$*: For any* (S, T), $(S', T') \in \ddot{\mathbb{S}}$,

-> *If* (S, T), (S', T'), $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$, *then* $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}1 \rangle$

-> *If* (S, T), (S', T'), $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$, *then* $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}2 \rangle$

Each element of $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ *is called a* relative dendrite.

In Fig. 3, two examples of two couples (S, T), $(S', T') \in \ddot{\mathbb{S}}$ which satisfy the conditions of $\langle \ddot{\mathrm{Z}}1 \rangle$ are given. Thus, in these two examples, $(S \cup S', T \cup T')$ is a relative dendrite.

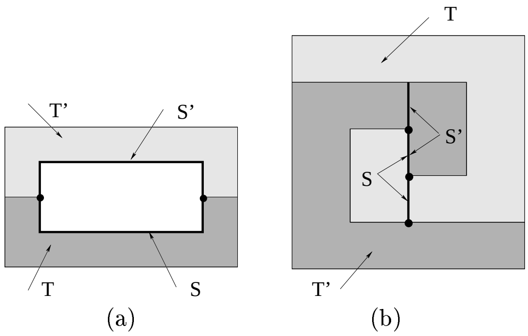

Fig. 3. (a) and (b): Two examples of two couples (S,T), $(S',T') \in \ddot{\mathbb{S}}$ which satisfy the conditions of $\langle \ddot{\mathrm{Z}}1 \rangle$ (we consider triangulations of these objects). In (a), S and S' are two simple open curves, $S \cap S'$ is a complex made of two vertices. In (b), S and S' are also two simple open curves, but $S \cap S'$ is a complex made of a segment.

In Fig. 3 (a), $(S \cup S', T \cup S')$ and $(T \cup S', T \cup T')$ are dyads (this fact may be seen using the forthcoming Prop. 9). Then, using $\langle \ddot{\mathrm{T}} \rangle$, it is possible to generate $(S \cup S', T \cup T')$ with the axioms of a dyad.
Now, we observe that, in Fig. 3 (b), $(S \cup S', T \cup S')$ is not a dyad (it can be checked that X and Y must have the same Euler characteristic whenever (X,Y) is a dyad). Thus, it is not possible to generate, in a straightforward manner, the relative dendrite $(S \cup S', T \cup T')$ with the axioms of a dyad.

Remark 2. As a direct consequence of the definitions of $\langle \ddot{\mathrm{Z}}1 \rangle$, $\langle \ddot{\mathrm{Z}}2 \rangle$, and the one of a dendrite, we have $\langle \ddot{\mathbb{C}}; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle \subseteq \{(X,Y) \in \ddot{\mathbb{S}} \mid X \in \mathbb{D}, Y \in \mathbb{D}\}$. This fact emphasizes the role of $(\{\emptyset\}, \{\emptyset\})$ in $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$.

Let $(X,Y) \in \ddot{\mathbb{S}}$. If α is a vertex such that $\alpha X \cap Y = X$, we say that $\alpha X \cup Y$ is a *cone on* (X,Y), and we write $\alpha X \ddot{\cup} Y$ for $\alpha X \cup Y$.

Proposition 4. *Let $Z \in \mathbb{S}$ and let α be an arbitrary vertex. There exists a unique couple $(X,Y) \in \ddot{\mathbb{S}}$ such that $Z = \alpha X \ddot{\cup} Y$.*

Thus, by Prop. 4, if $Z \in \mathbb{S}$ and if α is an arbitrary vertex, the complexes X and Y are specified whenever we write $Z = \alpha X \ddot{\cup} Y$. Note that we may have $\alpha \not\preceq Z$, in this case $X = \emptyset$ and $Z = Y$.

Proposition 5. *Let $Z, Z', Z'' \in \mathbb{S}$.*
We set $Z = \alpha X \ddot{\cup} Y$, $Z' = \alpha X' \ddot{\cup} Y'$, and $Z'' = \alpha X'' \ddot{\cup} Y''$.
1) If $Z = Z' \cup Z''$, then $X = X' \cup X''$ and $Y = Y' \cup Y''$;
2) If $Z = Z' \cap Z''$, then $X = X' \cap X''$ and $Y = Y' \cap Y''$.

Proof. The result follows from 1), 2), and Prop. 4.
1) If $Z = Z' \cup Z''$, then $Z = \alpha(X' \cup X'') \cup (Y' \cup Y'')$. Furthermore, since $(X' \cup X'') \subseteq (Y' \cup Y'')$ and since α is disjoint from $Y' \cup Y''$, we have $\alpha(X' \cup$

$X'') \cap (Y' \cup Y'') = X' \cup X''$.
2) Suppose $Z = Z' \cap Z''$. Then $Z = (\alpha X' \cup Y') \cap (\alpha X'' \cup Y'') = \alpha(X' \cap X'') \cup (Y' \cap Y'') \cup (\alpha X' \cap Y'') \cup (\alpha X'' \cap Y')$. Since $(\alpha X' \cap Y'') \cup (\alpha X'' \cap Y') \subseteq Y' \cap Y''$, we have $Z = \alpha(X' \cap X'') \cup (Y' \cap Y'')$. Furthermore, since $(X' \cap X'') \subseteq (Y' \cap Y'')$ and since α is disjoint from $Y' \cap Y''$, we have $\alpha(X' \cap X'') \cap (Y' \cap Y'') = X' \cap X''$. $\square$

Theorem 3. *Let $(X, Y) \in \ddot{\mathbb{S}}$. The couple (X, Y) is a relative dendrite if and only if $\alpha X \ddot{\cup} Y$ is a dendrite.*

Proof.
1) If $(X, Y) \in \ddot{\mathbb{C}}^+$, we see that $\alpha X \ddot{\cup} Y$ is a ramification. Thus, $\alpha X \ddot{\cup} Y$ is a dendrite. Suppose $R = \alpha S \ddot{\cup} T$ and $R' = \alpha S' \ddot{\cup} T'$ are dendrites. Then, by the very definition of a dendrite, $R \cap R'$ is a dendrite if and only if $R \cup R'$ is a dendrite. Consequently, by Prop. 5, $\alpha(S \cap S') \ddot{\cup} (T \cap T')$ is a dendrite if and only if $\alpha(S \cup S') \ddot{\cup} (T \cup T')$ is a dendrite. By the preceding remarks, we may affirm, by induction on $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$, that $\alpha X \ddot{\cup} Y$ is a dendrite whenever (X, Y) is a relative dendrite.
2) Suppose $Z = \alpha X \ddot{\cup} Y$ is a dendrite.
i) Suppose $Z \in \mathbb{C}$. If $X = \emptyset$, we have $(X, Y) \in \ddot{\mathbb{C}}$. If $X = \{\emptyset\}$, we must have $Y = \{\emptyset\}$, otherwise Z would not be connected, thus $(X, Y) \in \ddot{\mathbb{C}}^+$. If $X \neq \emptyset$ and $X \neq \{\emptyset\}$, it may be seen that we must have $X \in \mathbb{C}$ and $Y = X$, thus $(X, Y) \in \ddot{\mathbb{C}}$.
ii) Suppose we have $Z = Z' \cup Z''$, with $Z', Z'', Z' \cap Z'' \in \mathbb{D}$. We set $Z' = \alpha X' \ddot{\cup} Y'$ and $Z'' = \alpha X'' \ddot{\cup} Y''$. If (X', Y'), (X'', Y''), $(X' \cap X'', Y' \cap Y'')$ are relative dendrites, then $(X' \cup X'', Y' \cup Y'')$ is a relative dendrite (by $\langle \ddot{\mathrm{Z}}1 \rangle$), which means that (X, Y) is a relative dendrite (Prop. 5 (1)).
iii) Suppose we have $Z = Z' \cap Z''$, with $Z', Z'', Z' \cup Z'' \in \mathbb{D}$. We set $Z' = \alpha X' \ddot{\cup} Y'$ and $Z'' = \alpha X'' \ddot{\cup} Y''$. If (X', Y'), (X'', Y''), $(X' \cup X'', Y' \cup Y'')$ are relative dendrites, then $(X' \cap X'', Y' \cap Y'')$ is a relative dendrite (by $\langle \ddot{\mathrm{Z}}2 \rangle$), which means that (X, Y) is a relative dendrite (Prop. 5 (2)).
By i), ii), and iii), we may affirm, by induction on $\langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$, that (X, Y) is a relative dendrite whenever $\alpha X \ddot{\cup} Y$ is a dendrite. $\square$

7 Dyads and dendrites

The goal of this section is to derive a theorem (Th. 4) which makes clear the link between dyads and dendrites, this link is formulated with the notion of a relative dendrite. The proof of the theorem is made possible mainly thanks to the previous Th. 2 and 3 and the following propositions.

In the following proposition, and by the convention introduced in Section 2, the notation αA implicitly means that α is disjoint from the cell A. Thus, since $X \preceq Y \preceq A$, $\alpha X \cup Y$ is a cone on (X, Y) and $\alpha Y^*_A \cup X^*_A$ is a cone on (Y^*_A, X^*_A).

Proposition 6. *If $A \in \mathbb{C}$, and $X \preceq Y \preceq A$, then $(\alpha X \ddot{\cup} Y)^*_{\alpha A} = \alpha Y^*_A \ddot{\cup} X^*_A$.*

Proof. We have $(\alpha X \cup Y)^*_{\alpha A} = (\alpha X)^*_{\alpha A} \cap Y^*_{\alpha A}$. But $(\alpha X)^*_{\alpha A} = \alpha X^*_A$ (by Cor. 1 of [14]) and $Y^*_{\alpha A} = \alpha Y^*_A \cup A$ (by Cor. 2 of [14]). Thus, $(\alpha X \cup Y)^*_{\alpha A} = \alpha X^*_A \cap (\alpha Y^*_A \cup A) = \alpha Y^*_A \cup X^*_A$ (since $Y^*_A \preceq X^*_A$ and $X^*_A \preceq A$). $\square$

Proposition 7. *Let (X, Y) be a relative dendrite. We have $X \in \mathbb{D}$ if and only if $Y \in \mathbb{D}$.*

Proof. Let $(X, Y) \in \ddot{\mathbb{S}}$ such that $\alpha X \ddot{\cup} Y$ is a dendrite (see Th. 3).
i) Suppose $Y \in \mathbb{D}$. Since $\alpha X \in \mathbb{D}$ (Prop. 6 of [14]) then, by D2, we have $\alpha X \cap Y = X \in \mathbb{D}$.
ii) Let $A \in \mathbb{C}$ such that $Y \preceq A$, we suppose that α is disjoint from A. Suppose $X \in \mathbb{D}$. Thus, $X_A^* \in \mathbb{D}$ and $(\alpha X \cup Y)_{\alpha A}^* \in \mathbb{D}$ (Prop. 11 of [14]). But $(\alpha X \cup Y)_{\alpha A}^* = \alpha Y_A^* \cup X_A^*$ (Prop. 6). Since $\alpha Y_A^* \in \mathbb{D}$, by D2, it implies that $\alpha Y_A^* \cap X_A^* = Y_A^* \in \mathbb{D}$, thus $Y \in \mathbb{D}$ (Prop. 11 of [14]). $\square$

Lemma 1. *The collection $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ satisfies $\langle \ddot{\mathrm{Y}}1 \rangle$ and $\langle \ddot{\mathrm{Y}}2 \rangle$.*

Proof.
i) In $\langle \ddot{\mathrm{Z}}1 \rangle$, if we replace S by $S \cap T$, S' by S, and T' by S, we obtain:
-> If $(S \cap T, T)$, (S, S), $(S \cap T, S \cap T) \in \ddot{\mathcal{K}}$, then $(S, S \cup T) \in \ddot{\mathcal{K}}$.
ii) In $\langle \ddot{\mathrm{Z}}2 \rangle$, if we replace T by $S \cup T$, S' by T, and T' by T, we obtain:
-> If $(S, S \cup T)$, (T, T), $(S \cup T, S \cup T) \in \ddot{\mathcal{K}}$, then $(S \cap T, T) \in \ddot{\mathcal{K}}$.
iii) If $X \in \mathbb{S}$, then $\alpha X \ddot{\cup} X = \alpha X$ is a dendrite. Thus, by Th. 3, (X, X) is a relative dendrite. In consequence, if $\ddot{\mathcal{K}}$ is the collection made of all relative dendrites, we obtain $\langle \ddot{\mathrm{Y}}1 \rangle$ and $\langle \ddot{\mathrm{Y}}2 \rangle$ from i) and ii), respectively. $\square$

The following is easy to check.

Proposition 8. *Let $X, Y \in \mathbb{S}$.*
1) If $X, Y \in \mathbb{D}$, then $X \cap Y \in \mathbb{D}$ if and only if $X \cup Y \in \mathbb{D}$.
2) If $X, X \cap Y \in \mathbb{D}$, then $Y \in \mathbb{D}$ if and only if $X \cup Y \in \mathbb{D}$.
3) If $X, X \cup Y \in \mathbb{D}$, then $Y \in \mathbb{D}$ if and only if $X \cap Y \in \mathbb{D}$.
4) If $X \cap Y, X \cup Y \in \mathbb{D}$, then $X \in \mathbb{D}$ if and only if $Y \in \mathbb{D}$.

Lemma 2. *The collection $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ satisfies $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, and $\langle \ddot{\mathrm{L}} \rangle$.*

Proof. Let $(R, S) \in \ddot{\mathbb{S}}$ and $(S, T) \in \ddot{\mathbb{S}}$, and let α, β be two distinct vertices disjoint from T. Note that $\alpha R \cup S \preceq \alpha R \cup T$. We set $U = \beta(\alpha R \ddot{\cup} S) \ddot{\cup} (\alpha R \ddot{\cup} T)$.
i) We observe that $U = (\alpha\beta R) \cup (\beta S \cup T)$. We have $\alpha\beta R \in \mathbb{D}$, and $(\alpha\beta R) \cap (\beta S \cup T) = \beta R \in \mathbb{D}$. Thus, by Prop. 8 (2), $U \in \mathbb{D}$ if and only if $\beta S \cup T \in \mathbb{D}$, *i.e.*, if and only if (S, T) is a relative dendrite (Th. 3).
ii) Suppose (S, T) is a relative dendrite. By i) and Th. 3, $(\alpha R \ddot{\cup} S, \alpha R \ddot{\cup} T)$ is a relative dendrite. By Prop. 7, $\alpha R \ddot{\cup} S$ is a dendrite if and only if $\alpha R \ddot{\cup} T$ is a dendrite. By Th. 3, it follows that (R, S) is a relative dendrite if and only if (R, T) is a relative dendrite. This fact allows us to affirm that the collection $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ satisfies $\langle \ddot{\mathrm{T}} \rangle$ and $\langle \ddot{\mathrm{L}} \rangle$.
iii) Suppose that (R, S) and (R, T) are relative dendrites, thus $\alpha R \ddot{\cup} S$ and $\alpha R \ddot{\cup} T$ are dendrites (Th. 3). We have $U = \beta(\alpha R \ddot{\cup} S) \ddot{\cup} (\alpha R \ddot{\cup} T)$ and $\beta(\alpha R \ddot{\cup} S) \cap (\alpha R \ddot{\cup} T) = \alpha R \ddot{\cup} S$. Thus, we see that, by D1, the complex U is a dendrite. By i), it follows that (S, T) is a relative dendrite. This last fact allows us to affirm that the collection $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ satisfies $\langle \ddot{\mathrm{U}} \rangle$. $\square$

Lemma 3. *If $X \in \mathbb{D}$, then $(\emptyset, X) \in \ddot{\mathbb{X}}$.*

Proof.
i) If $X \in \mathbb{C}$, then $(\emptyset, X) \in \ddot{\mathbb{C}}$. Thus $(\emptyset, X) \in \ddot{\mathbb{X}}$.
ii) Let $S, T \in \mathbb{D}$ such that $S \cap T \in \mathbb{D}$. By $\langle \mathrm{D1} \rangle$, we have $S \cup T \in \mathbb{D}$. Suppose $(\emptyset, S)$, $(\emptyset, T)$, $(\emptyset, S \cap T) \in \ddot{\mathbb{X}}$. We have $(S \cap T, T) \in \ddot{\mathbb{X}}$ (Prop. 1 and $\langle \ddot{\mathrm{U}} \rangle$). Therefore $(S, S \cup T) \in \ddot{\mathbb{X}}$ (Th. 2 and $\langle \ddot{\mathrm{Y}}1 \rangle$). Then $(\emptyset, S \cup T) \in \ddot{\mathbb{X}}$ (Prop. 1 and $\langle \ddot{\mathrm{T}} \rangle$).
iii) Let $S, T \in \mathbb{D}$ such that $S \cup T \in \mathbb{D}$. By $\langle \mathrm{D2} \rangle$, we have $S \cap T \in \mathbb{D}$. Suppose $(\emptyset, S)$, $(\emptyset, T)$, $(\emptyset, S \cup T) \in \ddot{\mathbb{X}}$. We have $(S, S \cup T) \in \ddot{\mathbb{X}}$ (Prop. 1 and $\langle \ddot{\mathrm{U}} \rangle$). Therefore $(S \cap T, T) \in \ddot{\mathbb{X}}$ (Th. 2 and $\langle \ddot{\mathrm{Y}}2 \rangle$). Then $(\emptyset, S \cap T) \in \ddot{\mathbb{X}}$ (Prop. 1 and $\langle \ddot{\mathrm{L}} \rangle$).
By the very definition of a dendrite, the result follows by induction. $\square$

The following theorem is one of the main results of the paper. Intuitively, it asserts that, if (X, Y) is a dyad, then we cancel out all cycles of Y (*i.e.*, we obtain an acyclic complex), whenever we cancel out those of X (by the way of a cone, see Th. 3). Furthermore, Th. 4 asserts that, if we are able to cancel all cycles of Y by such a way, then (X, Y) is a dyad.

Theorem 4. *Let $(X, Y) \in \ddot{\mathbb{S}}$. We have $(X, Y) \in \ddot{\mathbb{X}}$ if and only if (X, Y) is a relative dendrite.*

Proof.
i) Suppose (X, Y) is a relative dendrite, *i.e.*, $(X, Y) \in \langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$. By Th. 3, we have $\alpha X \ddot{\cup} Y \in \mathbb{D}$ and, by Lemma 3, $(\emptyset, \alpha X \ddot{\cup} Y) \in \ddot{\mathbb{X}}$. We also have $(\emptyset, \alpha X) \in \ddot{\mathbb{X}}$ (Prop. 2). It means that $(\alpha X, \alpha X \ddot{\cup} Y) \in \ddot{\mathbb{X}}$ (Prop. 1 and $\langle \ddot{\mathrm{U}} \rangle$). We obtain $(\alpha X \cap Y, Y) = (X, Y) \in \ddot{\mathbb{X}}$ (Th. 2 and $\langle \ddot{\mathrm{Y}}2 \rangle$). Thus, $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle \subseteq \ddot{\mathbb{X}}$.
ii) The collection $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ contains $\ddot{\mathbb{C}}$ and satisfies $\langle \ddot{\mathrm{Y}}1 \rangle$, $\langle \ddot{\mathrm{Y}}2 \rangle$, $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, and $\langle \ddot{\mathrm{L}} \rangle$ (Lemmas 1 and 2). Thus $\langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle \subseteq \langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ (see remark 1). By Th. 2, the result is $\ddot{\mathbb{X}} \subseteq \langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$. $\square$

Trivially, we have $X \in \mathbb{D}$ if and only if $\alpha\emptyset \ddot{\cup} X \in \mathbb{D}$. Thus, by Th. 3, $X \in \mathbb{D}$ if and only if $(\emptyset, X)$ is a relative dendrite. It follows that, as a direct consequence of Th. 4, we have the following.

Corollary 1. *Let $X \in \mathbb{S}$. We have $X \in \mathbb{D}$ if and only if $(\emptyset, X) \in \ddot{\mathbb{X}}$.*

The following fact will be used for the illustration of the next section.

Proposition 9. *Let $X, Y, Z \in \mathbb{S}$ such that $X \preceq Y \preceq Z$.
If Y collapses onto X, then $(X, Z) \in \ddot{\mathbb{X}}$ if and only if $(Y, Z) \in \ddot{\mathbb{X}}$.
If Z collapses onto Y, then $(X, Y) \in \ddot{\mathbb{X}}$ if and only if $(X, Z) \in \ddot{\mathbb{X}}$.*

Proof. If Y collapses onto X, then it may be seen that $U' = \alpha Y \ddot{\cup} Z$ collapses onto $V' = \alpha X \ddot{\cup} Z$. Thus, U' is simple homotopic to V'. If Z collapses onto Y, then $U'' = \alpha X \ddot{\cup} Z$ collapses onto $V'' = \alpha X \ddot{\cup} Y$. Again, U'' is simple homotopic to V''. The result follows from Th. 3, Th. 4, and from Prop. 12 of [14]. This last proposition ensures that a complex S is a dendrite whenever it is simple homotopic to a dendrite. $\square$

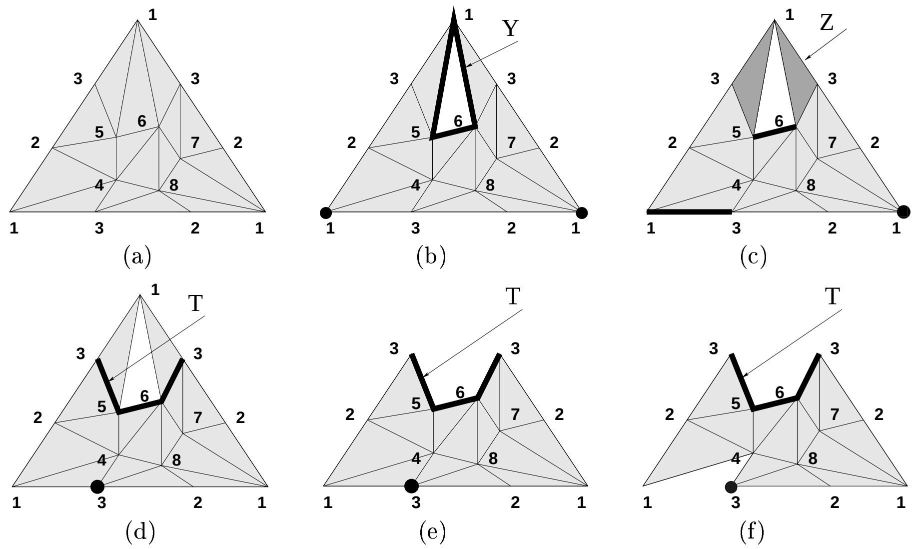

Fig. 4. (a): A triangulation D of the dunce hat, vertices with the same label have to be identified, (b): The complex $X = D \setminus \{1,5,6\}$ and the complex $Y \preceq X$ (highlighted), (c): The complex Z (highlighted) collapses onto Y, (d): The complex Z collapses onto T (highlighted), (e) and (f): The first steps of a collapse sequence of X onto T.

8 The dunce hat

We give, in this section, an illustration of the previous notions. We consider the complex D which is the triangulation of the dunce hat [16] depicted in Fig. 4 (a). As mentioned before, the dunce hat is contractible but not collapsible. In fact, it is possible to find a collapsible complex which collapses onto D (*e.g.*, see Th.1 of [16]). This shows that D is a dendrite. In the following, we will see that it is possible to recognize D as a dendrite without considering any complex larger than D (by using only "internal moves").

We consider the complex $X = D \setminus \{1,5,6\}$, we denote by C the cell whose facet is $\{1,5,6\}$, and by Y the complex $Y = C \cap X$, see Fig. 4 (b). We will see below that $(Y, X) \in \ddot{\mathbb{X}}$. By $\langle \ddot{\mathrm{Y}}1 \rangle$, this fact implies $(C, C \cup X) \in \ddot{\mathbb{X}}$, *i.e.*, $(C, D) \in \ddot{\mathbb{X}}$. Since $(\emptyset, C) \in \ddot{\mathbb{X}}$, by $\langle \ddot{\mathrm{T}} \rangle$, this implies $(\emptyset, D) \in \ddot{\mathbb{X}}$. Thus, by Cor. 1 of Th. 4, we get $D \in \mathbb{D}$. Now, we check that $(Y, X) \in \ddot{\mathbb{X}}$ using Prop. 9:
- The complex Z of Fig. 4 (c) collapses onto Y, thus $(Y, X) \in \ddot{\mathbb{X}}$ if $(Z, X) \in \ddot{\mathbb{X}}$;
- Z collapses onto the complex T of Fig. 4 (d), thus $(Z, X) \in \ddot{\mathbb{X}}$ if $(T, X) \in \ddot{\mathbb{X}}$;
- It could be checked that X collapses onto T, the first steps of a collapse sequence are given 4 (e) and (f). Thus, since $(T, T) \in \ddot{\mathbb{X}}$, we have $(T, X) \in \ddot{\mathbb{X}}$.

9 Conclusion

We introduced several axioms for describing dyads, *i.e.*, pair of complexes which are, in a certain sense, linked by a "relative topology". Our two main results are theorems 3 and 4 which make clear the links between dyads and dendrites, *i.e.*, between dyads and acyclic complexes.

We proposed an approach which is exclusively based on discrete notions and also, by the means of completions, on constructions on sets.

In the future, we will further investigate the possibility to develop a discrete framework related to combinatorial topology.

References

1. Whitehead, J.H.C.: Simplicial spaces, nuclei, and m-groups, Proc. London Math. Soc. (2), **45**, 243–327 (1939)
2. Björner, A.: Topological methods, Handbook of Combinatorics, R. Graham, M. Grötschel and L. Lovász (eds), North-Holland, Amsterdam, 1819–1872 (1995)
3. Hachimori, M.: Nonconstructible simplicial balls and a way of testing constructibility, Discrete Comp. Geom., Vol. 22, 223-230 (1999)
4. Kahn, J., Saks, M., Sturtevant, D.: A topological approach to evasiveness, Combinatorica 4, 297–306 (1984)
5. Welker, V.: Constructions preserving evasiveness and collapsibility, Discrete Math. 207, 243–255 (1999)
6. Jonsson, J.: Simplicial Complexes of Graphs, Springer Verlag (2008)
7. Kalai, G.: Enumeration of Q-acyclic simplicial complexes, Israel Journal of Mathematics, Vol. 45, No 4, 337–351 (1983)
8. Kong, T.Y.: Topology-preserving deletion of 1's from 2-, 3- and 4-dimensional binary images. LNCS 1347, Springer, 3–18 (1997)
9. Couprie, M., Bertrand, G.: New characterizations of simple points in 2D, 3D and 4D discrete spaces, IEEE Transactions on PAMI, 31 (4), 637–648 (2009)
10. Bertrand, G.: On critical kernels, Comptes Rendus de l'Académie des Sciences, Série Math. (345), 363–367, (2007)
11. Rosenfeld, A.: Digital topology, Amer. Math. Monthly, 621–630 (1979)
12. Kovalevsky, V. : Finite topology as applied to image analysis, Comp. Vision Graphics, and Im. Proc. 46, 141–161 (1989)
13. Kong, T.Y., Rosenfeld, A.: Digital topology: introduction and survey, Comp. Vision, Graphics and Image Proc. 48, 357–393 (1989)
14. Bertrand, G.: Completions and simplicial complexes, HAL-00761162 (2012)
15. Bing, R.H.: Some aspects of the topology of 3-manifolds related to the Poincaré Conjecture, Lectures on Modern Mathematics II, Wiley, 93–128 (1964)
16. Zeeman, E.C.: On the dunce hat, Topology Vol. 2, 341–358 (1964)
17. Serra, J.: Image Analysis and Mathematical Morphology, part II: theoretical advances, Academic Press, London (1988)
18. Aczel, P.: An introduction to inductive definitions, Handbook of Mathematical Logic, J. Barwise (ed.), 739-782(1977)